Influence of the interaction graph on the dynamics of Boolean networks

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Reference

This talk is based on the paper "On the influence of the interaction graph on a finite dynamical system," published in Natural Computing

Link:
https://link.springer.com/article/10.1007/s11047-019-09732-y

Finite dynamical systems

For $q \geq 2$, let $\llbracket q \rrbracket = \{0, 1, \ldots, q-1\}$.

For $n \geq 2$, let $[n] = \{1, \ldots, n\}$.

A Finite Dynamical System is a mapping in

$$\mathrm{F}(n, q) := \{f : \llbracket q \rrbracket^n \to \llbracket q \rrbracket^n\}.$$

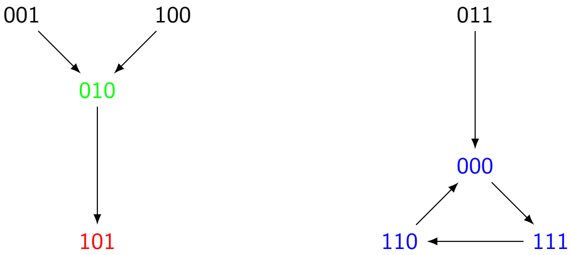
(For $q = 2$, we say f is a Boolean network.)

We view $x = (x_1, \ldots, x_n) \in \llbracket q \rrbracket^n$.

Similarly, we view $f = (f_1, \ldots, f_n)$, where $f_i : \llbracket q \rrbracket^n \to \llbracket q \rrbracket$.

Three dynamical properties

Let $f : \{0,1\}^3 \to \{0,1\}^3$ as follows.



Fixed points: $\mathrm{Fix}(f) = \{101\}$

Periodic points: $\mathrm{Per}(f) = \{000, 111, 110\} \cup \mathrm{Fix}(f)$

Images: $\mathrm{Ima}(f) = \{010\} \cup \mathrm{Per}(f)$

Interaction graph

The interaction graph of f, denoted $\mathbb{D}(f)$, has vertex set $[n]$ and uv is an arc in $\mathbb{D}(f)$ if and only if f_v depends essentially on x_u, i.e.

$$\exists a, b \in [\![q]\!]^n \text{ such that } a_{-u} = b_{-u}, f_v(a) \neq f_v(b).$$

For any graph $D = ([n], E)$ we denote the set of functions $f \in \mathrm{F}(n, q)$ with interaction graph either equal to D or to a subgraph of D as

$$\mathrm{F}[D, q] := \{f \in \mathrm{F}(n, q) : \mathbb{D}(f) = D\},$$
$$\mathrm{F}(D, q) := \{f \in \mathrm{F}(n, q) : \mathbb{D}(f) \subseteq D\}.$$

Influence of the interaction graph on the dynamics

We then consider $3 \times 3 \times 3$ quantities:

{ Minimum, Average, Maximum }

number of

{ Images, Periodic points, Fixed points }

in

$$\{\ \mathbf{F}_q, \mathrm{F}[D, q], \mathrm{F}(D, q)\ \}$$

Number of images

	$\mathbf{F}_q$	$\mathrm{F}[D, q]$	$\mathrm{F}(D, q)$
Min	1	Non-increasing function of q rank$^-$ reached for $q = (n+1)m$ Classification for rank$^- \in \{1, 2, 2^n\}$	1
Avg	$\sim (1 - e^{-1})q$	$\geq c_D\, q^{\alpha_1}$	$\geq c_D\, q^{\alpha_1}$
Max	q	q^{α_1} if $q \geq 3$ $= 2^n$ if $q = 2,\ D = K_n,\ n \neq 3$ $= 2^n$ if $q = 2,\ D = \vec{C}_n$ $< 2^n$ if $q = 2,\ D = C_n,\ n \geq 3$	q^{α_1}

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Number of periodic points

	$\mathbf{F}_q$	$\mathrm{F}[D, q]$	$\mathrm{F}(D, q)$
Min	1	1 if $q \geq 3$ 1 if $q = 2$ for many graphs D 2^n if $q = 2$ and $D = \vec{C}_n$	1
Avg	$\sim \sqrt{\pi q/2}$	1 if D is acyclic	1 if D is acyclic
Max	q	q^{α_n} if $q \geq 3$ $= 2^n$ if $q = 2$, $D = K_n$, $n \neq 3$ $= 2^n$ if $q = 2$, $D = \vec{C}_n$ $< 2^n$ if $q = 2$, $D = C_n$, $n \geq 3$	q^{α_n}

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Number of fixed points

	$\mathbf{F}_q$	$\mathrm{F}[D, q]$	$\mathrm{F}(D, q)$
Min	0	1 if D is acyclic 0 otherwise	1 if D is acyclic 0 otherwise
Avg	1	1	1
Max	q	$= q^\tau$ if $\tau \in \{0, 1\}$ $= q^{n-1}$ if $D = K_n$ $= \sum_k (q-1)^k I_k$ if $\tau = n$ $\geq \mathrm{fix}^+(D, q-1) + 1$ $\geq \nu + 1$ if $q = 2$ $\geq 2^{\nu'}$ if $q = 2$ $\geq q^{n - \mathrm{rank}_q(I-A)}$ $\geq \mathrm{mis}$	$= q^\tau$ if $\tau \in \{0, 1, 2, n-1, n\}$ $\leq A(n, q, \gamma)$ $\leq q^\tau$ $\leq q^H$ $\geq q^{n-\pi^*}$ $\geq q^{\nu^*}$ $\geq A(n, q, n-\delta+1)$ $\geq q^\delta / n$