Inverse problems on the blocksequential operator in Boolean networks

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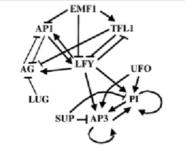
Motivation

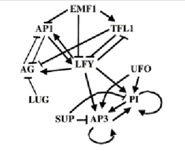
- Complex Systems



(L. Mendoza and E. Alvarez, 1998)

Motivation

- Complex Systems
- Boolean networks $f : \{0,1\}^n \to \{0,1\}^n$

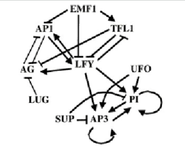


(L. Mendoza and E. Alvarez, 1998)

$$
\begin{aligned}
f_1(x) &= x_4 \\
f_2(x) &= x_1 \wedge x_2 \\
f_3(x) &= x_2 \vee x_3 \\
f_4(x) &= x_3 \wedge x_4
\end{aligned}
$$

Motivation

- Complex Systems
- Boolean networks $f : \{0,1\}^n \to \{0,1\}^n$
  - Interaction Graph



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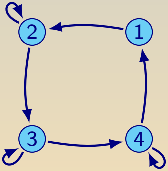


Block-sequential schedule

Definition

A *block-sequential schedule* is an ordered partition of the components of a Boolean network which defines the order in which the states of the network are updated in one unit of time.

Examples

$$s_1 = \{3,4\}\{1\}\{2\},$$

$$s_2 = \{1,2,3,4\},$$

$$s_3 = \{2\}\{3\}\{4\}\{1\}.$$

Labeled digraph

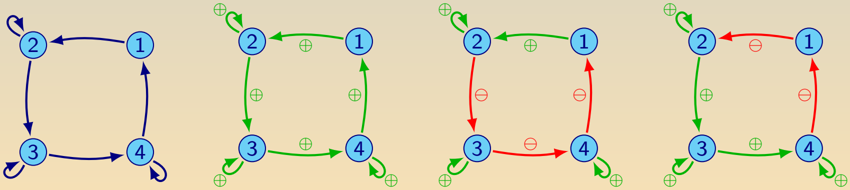
Given a interaction graph G and a block-sequential schedule s, a labeled digraph (G, s) is a digraph with a labeling function lab_s:

$$lab_s : A(G) \to \{\oplus, \ominus\}$$

$$lab_s(u, v) = \oplus \iff s(u) \geq s(v)$$

G $\quad$ $s_1 = \{1, 2, 3, 4\}$ $\quad$ $s_2 = \{2\}\{3\}\{4\}\{1\}$ $\quad$ $s_3 = \{3, 4\}\{1\}\{2\}$

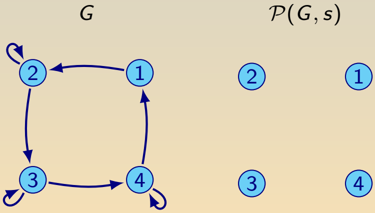
$s_4 = \{4\}\{1\}\{3, 2\}$



Parallel digraph

The *parallel digraph* is a digraph that represent the real dependence of the components of the Boolean network, according with the block-sequential schedule.
Also, is equivalent to the interaction graph of a Boolean network with equal dynamic behavior when is updated in parallel (one only block).

$s = \{2\}\{3\}\{4\}\{1\}$



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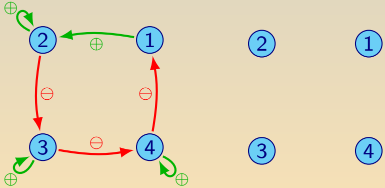
(G, s) $\mathcal{P}(G, s)$



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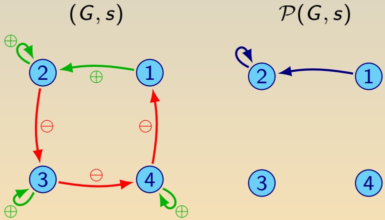

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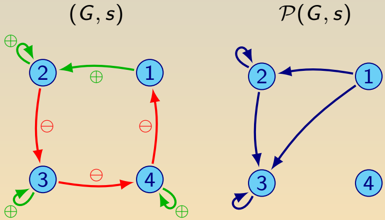

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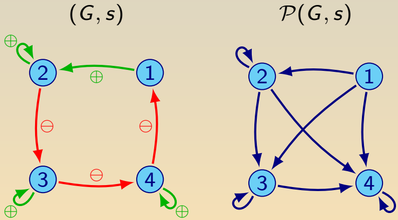

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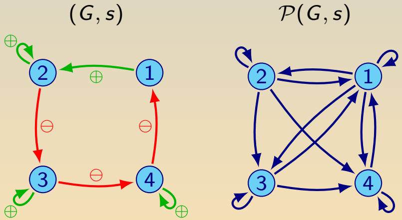
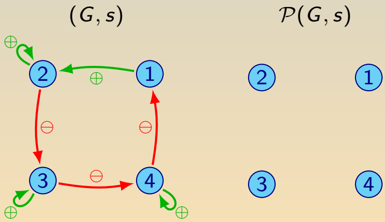


Parallel digraph

Can be obtained from the labeled digraph.
$\forall(u,v) \in V(G) \times V(G), (u,v) \in A(\mathcal{P}(G,s))$ if and only if:

$s = \{2\}\{3\}\{4\}\{1\}$



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$\forall (u,v) \in V(G) \times V(G), (u,v) \in A(\mathcal{P}(G,s))$ if and only if:

- (u,v) is labeled $\oplus$.

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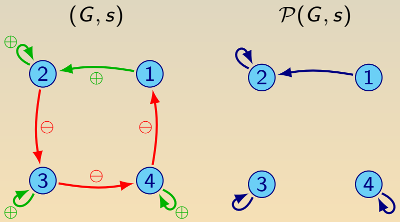

Parallel digraph

Can be obtained from the labeled digraph.
$\forall (u,v) \in V(G) \times V(G), (u,v) \in A(\mathcal{P}(G,s))$ if and only if:

- (u,v) is labeled $\oplus$.
- $\exists w \in V(G), (u,w)$ is labeled $\oplus$ and exists a path from w to v labeled $\ominus$.

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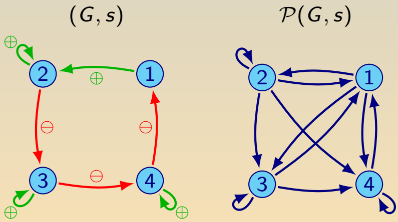
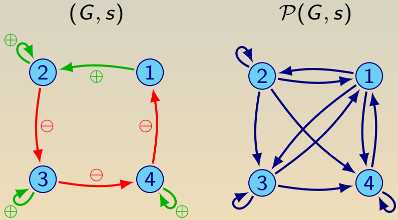


Parallel digraph

The function that associates (G, s) with the digraph $\mathcal{P}(G, s)$ is called **block-sequential operator** and can be constructed in polynomial time.

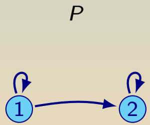
$s = \{2\}\{3\}\{4\}\{1\}$



Inverse problem

Given a digraph P and a block-sequential schedule s, does there exist a digraph G such that $\mathcal{P}(G, s) = P$?

$$s = \{1\}\{2\}$$



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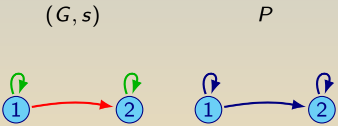

Unique solution

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Given a digraph P and a block-sequential schedule s, does there exist a digraph G such that $\mathcal{P}(G, s) = P$?

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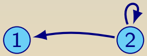
P



Inverse problem

Given a digraph P and a block-sequential schedule s, does there exist a digraph G such that $\mathcal{P}(G,s) = P$?

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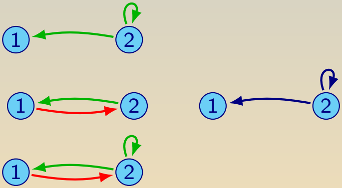
(G,s) $\quad$ P

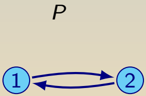


Multiple solutions

Inverse problem

Given a digraph P and a block-sequential schedule s, does there exist a digraph G such that $\mathcal{P}(G, s) = P$?

$s = \{1\}\{2\}$



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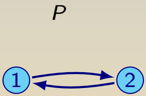

No solution

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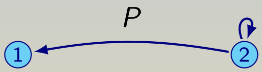
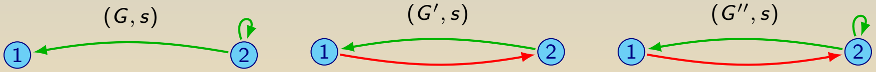
Results



For example, for the schedule $\{1\}\,\{2\}$, there are three preimages:



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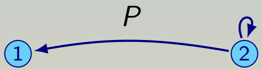

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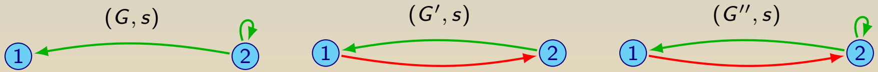
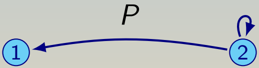


Theorem

Let s be a block-sequential schedule and G and G' two digraphs such that $\mathcal{P}(G,s) = \mathcal{P}(G',s)$. Then $\mathcal{P}(G \cup G', s) = \mathcal{P}(G,s)$.

Results



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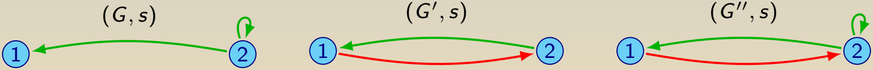


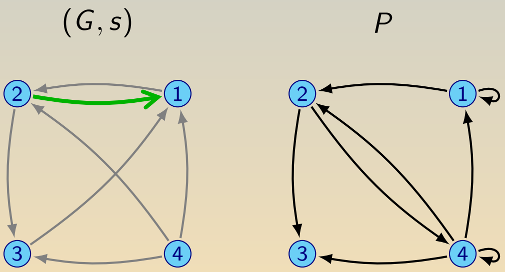
Theorem

For this reason, if for a digraph P and a block-sequential schedule s there exists at least one preimage G, then there exists a maximum preimage.

Rules

$\overline{P}$ Rule

$\forall (u, v)$ such that $lab(u, v) = \oplus$, if $(u, v) \notin P$, then $(u, v) \notin G$.



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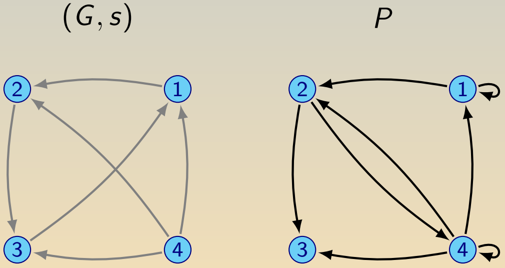
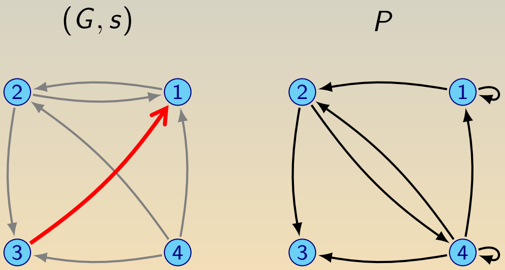


Rules

Transitive Rule

$\forall(u,v)$ such that $lab(u,v) = \ominus$, if $\exists w$ such that $(w,u) \in P$ and $(w,v) \notin P$, then $(u,v) \notin G$.



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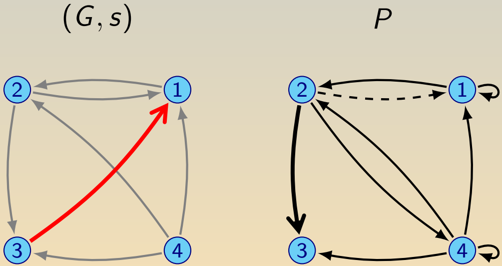

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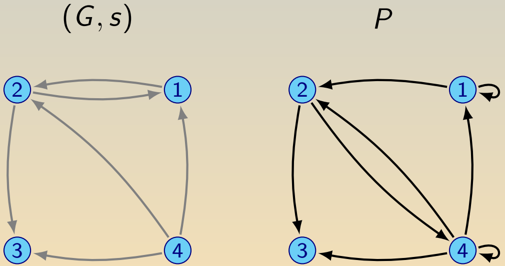
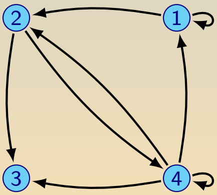


Algorithm MaxPI: Step 1 - Build and label

Input

Given a digraph P and a block-sequential schedule $s = \{3\}\{1\}\{2,4\}$.

P



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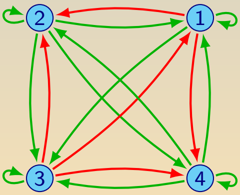
(G, s)



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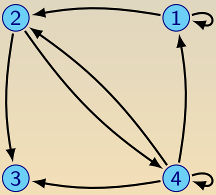
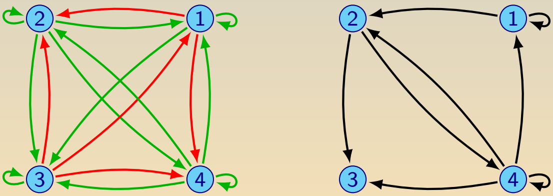


Algorithm MaxPI: Step 2a
Removing green arcs

Rule

$\forall (u, v) \in A(G)$ that does not satisfy the "$\overline{P}$ rule",
(u, v) is removed from G.

(G, s) P



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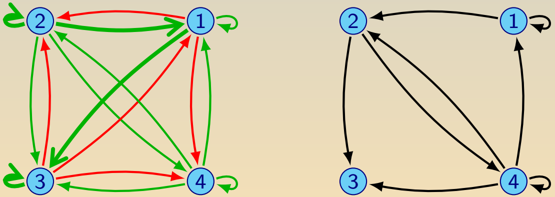

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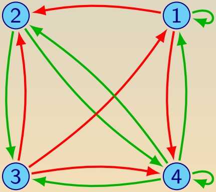
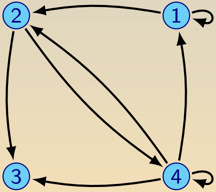


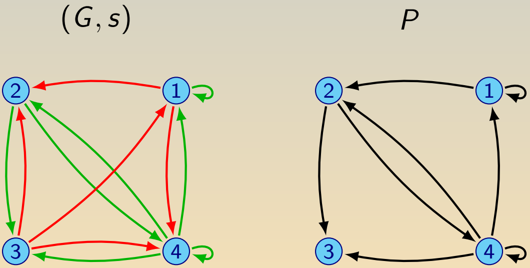
P



Algorithm MaxPI: Step 2b
Removing red arcs

Rule

$\forall(u,v) \in A(G)$ that does not satisfy the "Transitive rule", (u,v) is removed from G.



Algorithm MaxPI: Step 2b
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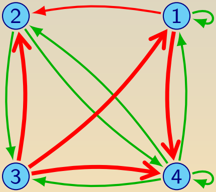

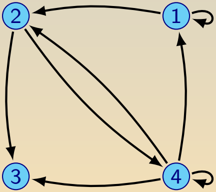

Algorithm MaxPI: Step 2b Removing red arcs

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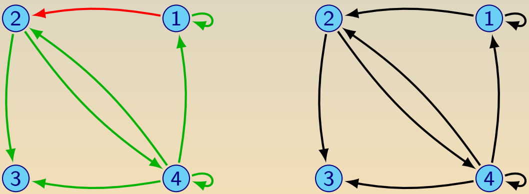
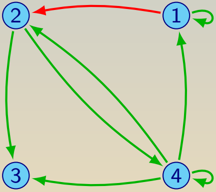
(G, s) $\quad\quad P$

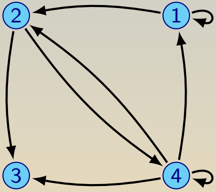


Algorithm MaxPI: Step 3 - Validation

(G, s)



P



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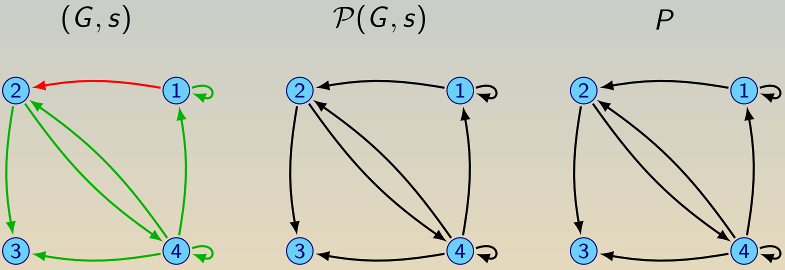

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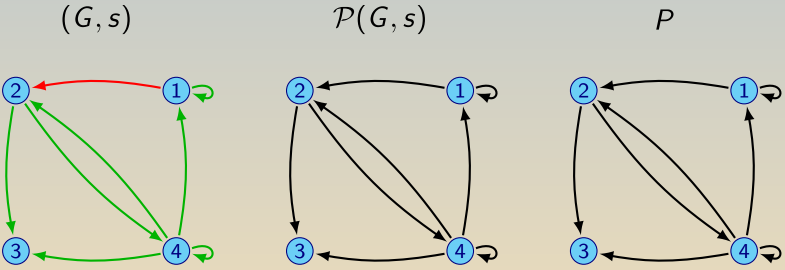


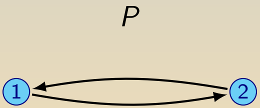
Output

Since $\mathcal{P}(G, s) = P$, the algorithm return (G, s) as maximun preimage of P with the schedule s.

Another example of MaxPI: Step 1

Input

Given a digraph P and a block-sequential schedule $s = \{1\}\{2\}$.



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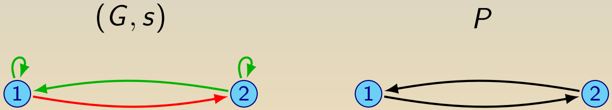
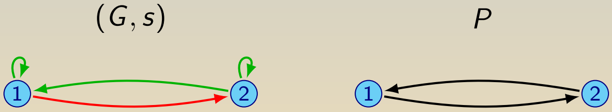
Initially: $G \leftarrow K_n,\ \ n = |V(P)|$.



Another example of MaxPI: Step 2a

Removing green arcs, according "$\overline{P}$ rule".



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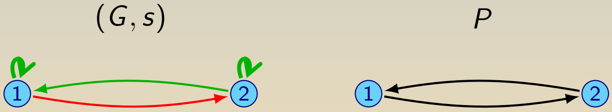

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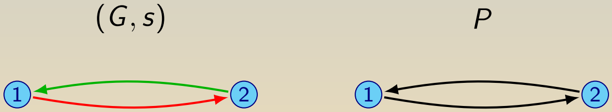

Another example of MaxPI: Step 2b

Removing red arcs, according "Transitive rule".

Another example of MaxPI: Step 2b

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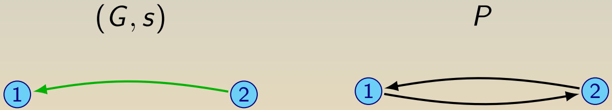
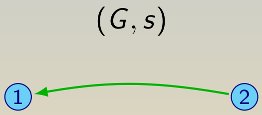
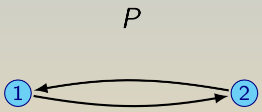


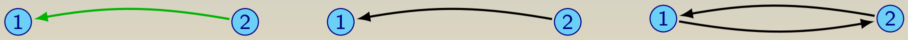
Another example of MaxPI: Step 3





Another example of MaxPI: Step 3

(G, s) $\mathcal{P}(G, s)$ P



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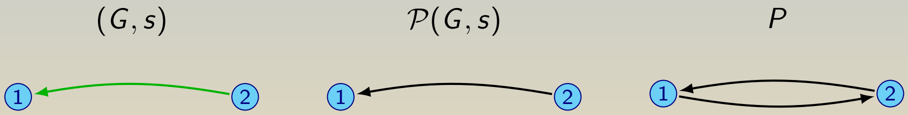
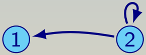


Output

Since $\mathcal{P}(G,s) \neq P$, P does not have preimage for the schedule s.

Algorithm for enumeration of preimages

P $\qquad$ G_{MaxPI} $\qquad$ $s = \{1\}\{2\}$



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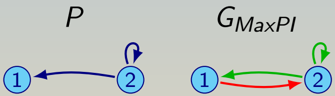

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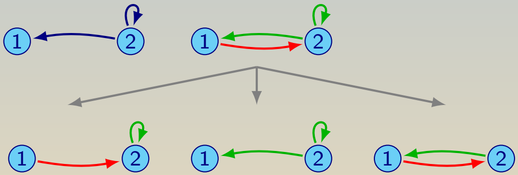

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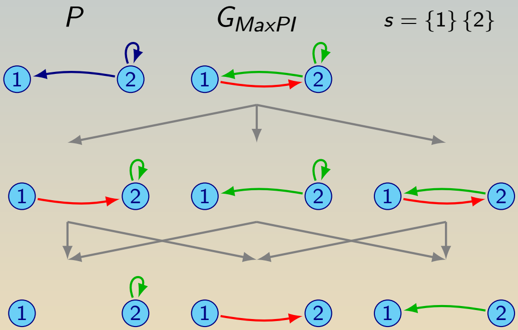

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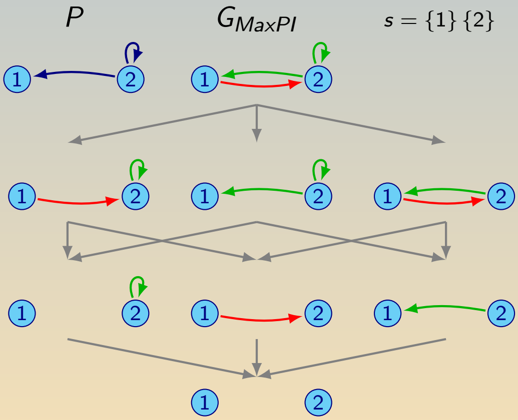

Algorithm for enumeration of preimages

Lemma

Let s be a block-sequential schedule and G and G' two digraphs such that $V(G) = V(G')$. If $G \subseteq G'$, then $\mathcal{P}(G, s) \subseteq \mathcal{P}(G', s)$.

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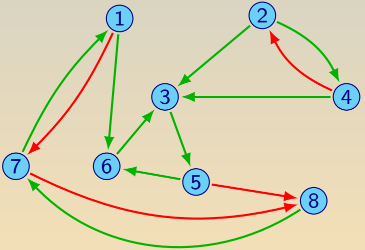
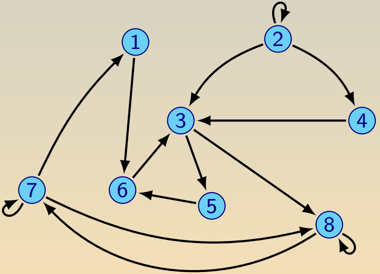
(G, s)



$\mathcal{P}(G, s)$



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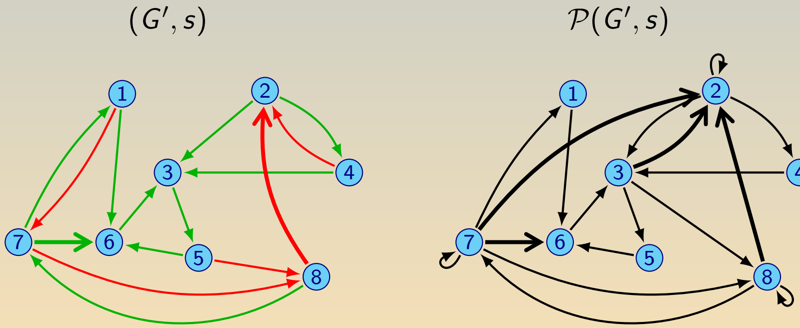

Algorithm for enumeration of preimages

Proposition

Let s be a block-sequential schedule and G and G'' two digraphs such that $G'' \subseteq G$. If $\mathcal{P}(G, s) = \mathcal{P}(G'', s)$, then
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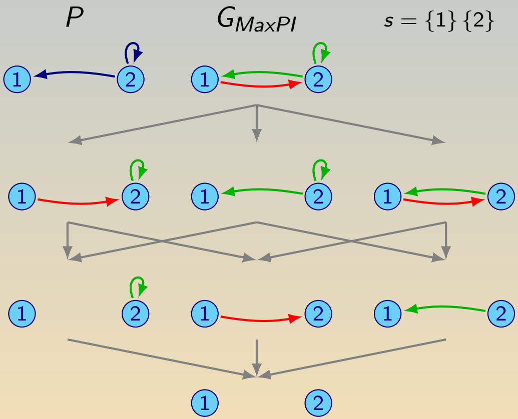
Proof

Since $G'' \subseteq G' \subseteq G$, then $\mathcal{P}(G'', s) \subseteq \mathcal{P}(G', s) \subseteq \mathcal{P}(G, s)$. Since $\mathcal{P}(G'', s) = \mathcal{P}(G, s)$, then $\mathcal{P}(G'', s) = \mathcal{P}(G', s) = \mathcal{P}(G, s)$.

Algorithm for enumeration of preimages



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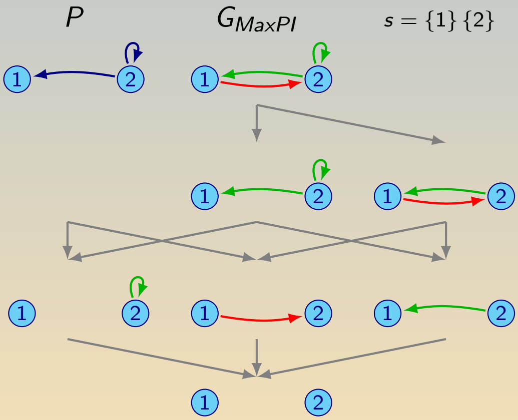

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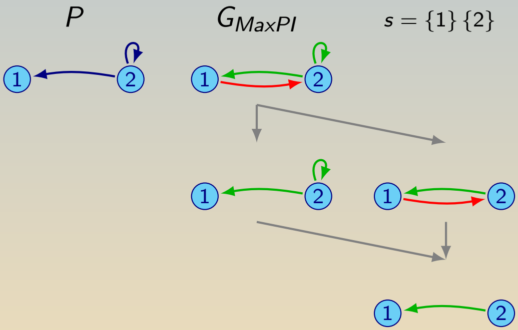

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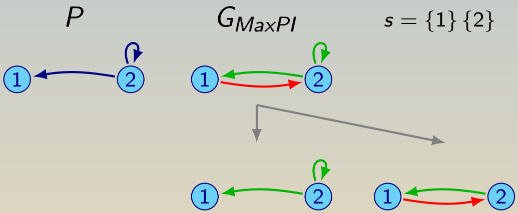

Algorithm for enumeration of preimages

Complexity

The complexity of this algorithm has a **polynomial delay**.

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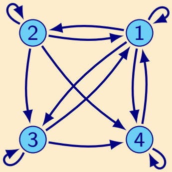
Since there are cases with an exponential number of pre-images, listing all the pre-images has an **exponential cost**
For example, this digraph with the block-sequential schedule $\{2\}\{3\}\{4\}\{1\}$ has 8 preimages, corresponding to $2^{\frac{(n-2)(n-1)}{2}}$.

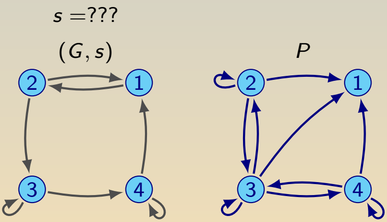


Contents

Work in progress...

Problem

Given two digraphs G and P, does there exist a block-sequential schedule s such that $\mathcal{P}(G, s) = P$?



Work in progress...

Theorem

If G and P are acyclic digraphs, it is possible to decide if there is labeling function lab such that $\mathcal{P}(G, lab) = P$ in polynomial time.

Work in progress...

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Work in progress...

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With an algorithm that label the arcs of G, according the following rules:

- "Transitive rule": If $\exists u, v, w \in V(G)$, such that $(u, v) \in G, (w, u) \in P$ and $(w, v) \notin P$, then $lab(u, v) = \oplus$.

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Work in progress...

If one arc is labeled $\oplus$ and $\ominus$ by different rules, then the decision problem answer is "There is no labeling function *lab* such that $\mathcal{P}(G, lab) = P$"..

Work in progress...

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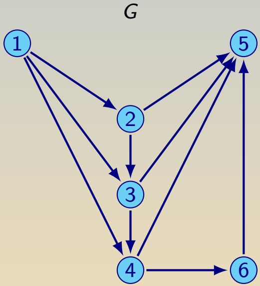
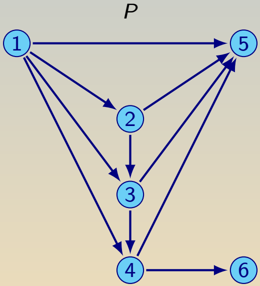
Otherwise, the labeling function formed by all the arcs labeled by the algorithm plus negative arcs (replacing the arcs not labeled by the algorithm) is a labeling function such that $\mathcal{P}(G, lab) = P$.

Work in progress...





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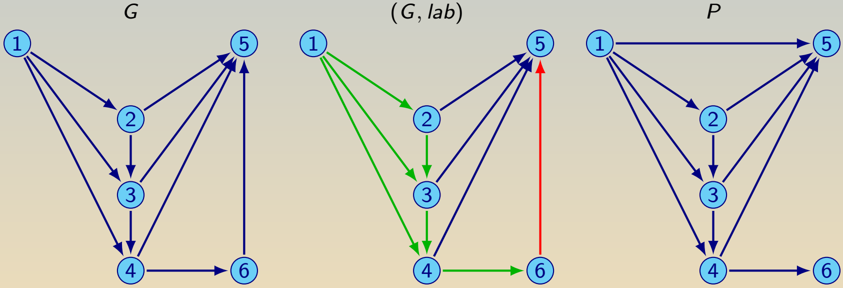

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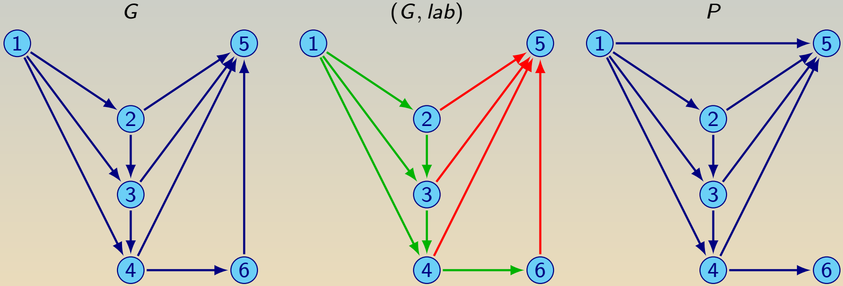

Work in progress...

- And for the acyclic case?

Work in progress...

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Still in progress

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Still in progress

Thank You!