Boolean network classes

Maximilien Gadouleau
Including joint work with Florian Bridoux and Guillaume Theyssier

IWBN2020, Concepción, January 2020

Outline

Outline

Boolean networks

A Boolean network (BN) is any

$$f : \{0,1\}^n \to \{0,1\}^n.$$

We see $f = (f_1, \ldots, f_n)$, where each $f_v : \{0,1\}^n \to \{0,1\}$ is a Boolean function.

Similarly, we see $x = (x_1, \ldots, x_n) \in \{0,1\}^n$.

The need for BN classes

It's typical in maths to consider classes of objects with special properties.

Examples for graphs: trees, bipartite graphs, cographs, chordal graphs, perfect graphs, interval graphs, etc.

There are a lot of BNs! Here are the number of different objects on a set of n elements:

- (Simple) graphs: $2^{\binom{n}{2}}$
- Digraphs, a.k.a. binary relations, a.k.a. Boolean matrices: 2^{n^2}
- Hypergraphs, a.k.a. set families, a.k.a. Boolean functions: 2^{2^n}
- Boolean networks: $\left(2^{2^n}\right)^n = 2^{n2^n}$.

Therefore, we need to look at Boolean network classes.

Outline

BN classes: interaction graph

The interaction graph of f, denoted $\mathbb{D}(f)$, has vertex set $[n]$ and uv is an arc in $\mathbb{D}(f)$ if and only if f_v depends essentially on x_u, i.e.

$$\exists a, b \in \{0,1\}^n \text{ such that } a_{-u} = b_{-u}, f_v(a) \neq f_v(b).$$

Seminal result: Robert's theorem. (Robert 80)
If $\mathbb{D}(f)$ is acyclic, then f has a unique, globally attractive fixed point ($f^n(x) = c$ for all x).

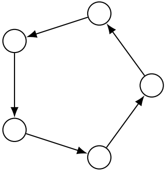
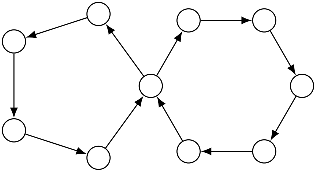
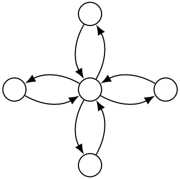
BN classes: interaction graph

Extensions of Robert's theorem.

- Signed version: no positive cycles, no negative cycles (Aracena 04; Richard 10)
- Quantitative version: (Positive) feedback bound (Aracena 08; Riis 07) and many results after that
- Complexity results in the signed case (Bridoux, Dubec, Perrot, Richard 19)
- Dynamic characterisation of BNs with acyclic interaction graphs (G 20+)

BN classes: interaction graph

Other results for the following interaction graphs:

- Cycles (Remy, Mossé, Chaouiya, Thieffry 03;

Demongeot, Sené, Noual 12)

- Double cycles (Noual 10)

- Flower graphs (Didier, Remy 12)

BN classes: local functions

There is a natural partial order on $\{0,1\}^n$: $x \leq y$ if $x_i \leq y_i$ for all $1 \leq i \leq n$.

f is monotone if $x \leq y \implies f(x) \leq f(y)$. Equivalently, f is monotone if $x \leq y \implies f_i(x) \leq f_i(y)$ for all $1 \leq i \leq n$.

Seminal result: Knaster-Tarski theorem. (Knaster 28; Tarski 55)
If f is monotone, then $\mathrm{Fix}(f)$ is a lattice (and hence, is not empty).

Related results:

- Bounds on the number of fixed points in (Aracena, Richard, Salinas 17)
- Fixed points asynchronously reachable by a geodesic (Richard 10; Melliti, Regnault, Richard, Sené 13)
- Monotone networks are fixable in cubic time (Aracena, G, Richard, Salinas, 20+)

BN classes: local functions

Further results for other classes based on local functions:

- Monotone conjunctive networks (AND-networks)

  $$f_i(x) = \bigwedge_{j \in N(i)} x_j$$

  are fixable in linear time
- Number of fixed points of conjunctive networks

  $$f_i(x) = \bigwedge_{j \in N^+(i)} x_j \wedge \bigwedge_{k \in N^-(i)} \bar{x}_k$$

  (Aracena, Demongeot, Goles 04; Aracena, Richard, Salinas 14)
- Goles's theorem on symmetric threshold networks (Goles 80 and many extensions): period at most 2 on parallel, only fixed points in sequential
- Linear networks: see linear algebra
- Majority function, freezing networks: ask Eric

Outline

BN classes: metric properties

There is a natural metric on $\{0,1\}^n$, namely the Hamming metric

Seminal result. (Polya 40)
The following are equivalent:

1. f is an isometry (i.e. $d_\mathrm{H}(f(x), f(y)) = d_\mathrm{H}(x, y)$)
2. f is an automorphism of the hypercube (i.e. f is bijective and $d_\mathrm{H}(x, y) = 1$ implies $d_\mathrm{H}(f(x), f(y)) = 1$)
3. f is a union of cycles.

Extension to non-expansive networks, where $d_\mathrm{H}(f(x), f(y)) \leq d_\mathrm{H}(x, y)$ (i.e. it is 1-Lipschitz) (Feder 92):

- Characterisation of sets of fixed points of non-expansive networks
- Dynamics are ultimately those of an isometry

BN classes: asynchronous properties

Let $b \subseteq [n]$, then

$$f^{(b)}(x) = (f_b(x), x_{[n]\setminus b}).$$

For any word $w = (w_1, \dots, w_t)$ with $w_i \subseteq [n]$, we denote

$$f^w = f^{(w_t)} \circ \cdots \circ f^{(w_1)}.$$

A word $B = (b_1, \dots, b_t)$ is block-sequential if $b_i \cap b_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^t b_i = [n]$.

Proposition. (Bridoux, G, Theyssier, "Commutative automata networks")
The following are equivalent.

1. f is commutative, i.e. $f^{(i,j)} = f^{(j,i)}$ for all $i, j \in [n]$
2. $f^{(b,c)} = f^{(c,b)}$ for all $b, c \subseteq [n]$
3. $f^B = f^C$ for any two block-sequential words B, C of $[n]$
4. $f = f^B$ for any block-sequential word B of $[n]$.

In other words, commutative networks are robust to changes in the update schedule.

BN classes: asynchronous properties

Theorem. (Bridoux, G, Theyssier, "Commutative automata networks")
A Boolean network is commutative if and only if it is a union of arrangement networks.

We define arrangements as follows.

A subcube of $\{0,1\}^n$ is any set of the form $\mathrm{X}[s,\alpha] := \{x \in \{0,1\}^n, x_s = \alpha\}$ for some $s \subseteq Z$ and $\alpha \in \{0,1\}^s$.
A family of subcubes $X = \{X_\omega : \omega \in \Omega\}$ is called an arrangement if $X_\omega \cap X_\xi \neq \emptyset$ for all $\omega, \xi \in \Omega$ and $X_\omega \not\subseteq X_\xi$ for all $\omega \neq \xi$.
We denote the content of X by $\hat{X} := \bigcup_{\omega \in \Omega} X_\omega$.

If $X = \{X_\omega = \mathrm{X}[s^\omega, \alpha^\omega] : \omega \in \Omega\}$ is an arrangement, then the dimensions of $\hat{X}$ are as follows.

- Let $\tau := \bigcap_{\omega \in \Omega} s^\omega$, then τ is the set of external dimensions of $\hat{X}$.
- Let $\sigma := \bigcup_{\omega \in \Omega} s^\omega$, then $[n] \setminus \sigma$ is the set of free dimensions of $\hat{X}$. Then $\bigcap_{\omega \in \Omega} X_\omega = \mathrm{X}[\sigma, \alpha]$.
- The other dimensions in $\sigma \setminus \tau$ are the tight dimensions of $\hat{X}$.

Arrangement network: Let X be an arrangement. Then on $\hat{X}$, let

1. $f_i(x) = \alpha_i$ for every tight dimension i of $\hat{X}$,
2. f_j be uniform nontrivial for any free dimension j,
3. and f_k be trivial on any external dimension of $\hat{X}$.

Outside of $\hat{X}$, f is trivial: $f(x) = x$ if $x \notin \hat{X}$.

Any arrangement network is commutative.

We can combine families of commutative networks as follows.

x is an unreachable fixed point of f if

$$f^{(s)}(y) = x \iff y = x \qquad \forall s \subseteq [n], s \neq \emptyset.$$

Let $R(f)$ be the set of non-(unreachable fixed points) of f. If $\{f^a : a \in A\}$ is a family of networks with $R(f^a) \cap R(f^{a'}) = \emptyset$ for all $a, a' \in A$, we define their union as

$$F(x) := \bigcup_{a \in A} f^a(x) = \begin{cases} f^a(x) & \text{if } x \in R(f^a) \\ x & \text{otherwise.} \end{cases}$$

Any union of arrangement networks is commutative.

BN classes: looking further

Some other ways of defining BN classes:

- Recursively
- Substructure definition of BN: subnetwork, reduction, Boolean derivative...
- Finite field form: f is a polynomial over $\mathrm{GF}(2^n)$ ($f(x) = \alpha x$ used in (Bridoux, G, Theyssier 20+))
- Using clones for families of local functions (see Post's lattice)

Merci !

¡Muchas gracias!

Thank you!