Combinatorics and dynamical classification of an Ising cellular automaton: the Q2R model

Marco Montalva Medel
Universidad Adolfo Ibáñez

[Joint work with Sergio Rica (UAI) and Felipe Urbina (U. Mayor)]

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Outline

- The Q2R model definition and its main properties.
- Preliminary results.
- Partition of the set of configurations in the Q2R dynamic.
- Classification of limit cycles in Q2R.
- Combinatorial and dynamical results on the Q2R.
- General overview of the Q2R dynamics.
- Exhaustive study of the (4×4) case.
- Discussion.

The Q2R[2] model

- Introduced by Vichniac[1] in the mid-80's, it is a cellular automata representation of the two-dimensional Ising model for ferromagnetism that possesses quite a rich and complex dynamics.
- It has the property of being reversible, i.e., any configuration in its dynamics belongs to an attractor (fixed point or limit cycle).
- It has the property of being conservative; there are different energy functions which are invariants under the Q2R dynamics.
- It is defined in a regular two dimensional toroidal lattice with even rank $L \times L$, being $N = L^2$ the total number of *nodes*.

[1] G. Vichniac, *Simulating Physics with Cellular Automata.* Physica **D 10** (1984), 96–116.

[2] "Q" by *quatre* (four, in french) and represents the number of neighbors, "2" by the *two-step* dynamic, and "R" by the fact of being *reversible*.

The Q2R model

- The nodes have associated an index $\boldsymbol{k} \in \{1, \ldots, N\}$, as well as a relative position in the lattice specified by two indices $k_1 \in \{1, \ldots, L\}$ and $k_2 \in \{1, \ldots, L\}$ (the respective row and column indices).
- A node $\boldsymbol{k}$ is characterized by two possible values $x_{\boldsymbol{k}} = \pm 1$, conforming with the following two-step rule:

$$x_{\boldsymbol{k}}^{t+1} = x_{\boldsymbol{k}}^{t-1} H\left(\sum_{\boldsymbol{i} \in V_{\boldsymbol{k}}} x_{\boldsymbol{i}}^{t}\right)$$

  where:
  - $V_{\boldsymbol{k}}$ denotes the von Neumann neighborhood of its four closest neighbors, with periodic boundary conditions.
  - The function H is such a that $H(s = 0) = -1$ and $H(s) = +1$ in all other cases.
  - It requires two initial conditions, x^0 and x^1, in order to start its dynamic and to obtain in the next *time step transition* x^2, and so forth.

The Q2R model

- The **state** x^t belongs to the discrete set $\Omega \equiv \{-1,1\}^N$ (of size 2^N).
- The set of **configurations**, denoted by Ω^2, it is composed by couples of states in $\Omega^2 = \Omega \times \Omega = \{(x,y) \mid x \in \Omega \wedge y \in \Omega\}$ (of size 2^{2N}).
- We rewrite the above two-step rule as the following one-step rule:

$$
\begin{aligned}
y^{t+1} &= x^t \\
x^{t+1} &= y^t \odot \phi\left(x^t\right)
\end{aligned}
$$

where:
  - The symbol $\odot$ denotes the Hadamard product (multiplication component to component for two matrices).
  - $\phi : \Omega \to \Omega$ is the function such that, $[\phi(x)]_{\boldsymbol{k}} = -1 \Leftrightarrow \sum_{\boldsymbol{i} \in V_{\boldsymbol{k}}} x_{\boldsymbol{i}} = 0$, i.e., if the sum of all von Neumann neighbors of the $\boldsymbol{k}$-th node is null. Otherwise, $[\phi(x)]_{\boldsymbol{k}} = +1$.

Example of a $\phi(x)$ calculation

$$\underbrace{\begin{bmatrix} \boxed{\boxed{1}} & 1 & \boxed{\boxed{1}} & 1 \\ 1 & \boxed{-1} & 1 & \boxed{\boxed{1}} \\ \boxed{1} & 1 & \boxed{-1} & 1 \\ 1 & \boxed{1} & 1 & \boxed{\boxed{1}} \end{bmatrix}}_{x} \mapsto \underbrace{\begin{bmatrix} 4 & 2 & 4 & \boxed{\boxed{4}} \\ 2 & 4 & 0 & 4 \\ 4 & \boxed{0} & 4 & 2 \\ 4 & 4 & 2 & 4 \end{bmatrix}}_{\mu(x)} \mapsto \underbrace{\begin{bmatrix} 1 & 1 & 1 & \boxed{\boxed{1}} \\ 1 & 1 & -1 & 1 \\ 1 & \boxed{-1} & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}}_{\phi(x)}$$

Remark

The state x does not have any null-neighborhood iff $\phi(x) = \mathbb{1}$. Notice that $\phi(\mathbb{1}) = \phi(-\mathbb{1}) = \mathbb{1}$, where $\mathbb{1}, -\mathbb{1} \in \Omega$ are the states composed only by 1s and -1s, respectively.

Next, we show an example of some time step transitions in the Q2R dynamic by using the one-step rule.

$$(x^0, y^0) = \left(\begin{bmatrix} -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \right)$$

$$(x^1, y^1) = \left(\begin{bmatrix} -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \end{bmatrix} \right)$$

$$(x^2, y^2) = \left(\begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 \end{bmatrix} \right)$$

where $x^1 = y^0 \odot \phi(x^0)$, $y^1 = x^0$, $x^2 = y^1 \odot \phi(x^1)$, $y^2 = x^1$,
$x^3 = y^2 \odot \phi(x^2) = x^0$, $y^3 = x^2 = y^0$.

Notation

$(x^0, y^0) \to (x^1, y^1) \to (x^2, y^2) \to (x^0, y^0)$, *or simply:*
$(x, y) \to (z, x) \to (y, z) \to (x, y)$

Partitioning Ω^2

Firstly, we consider $\Omega^2 = \Omega^2_{xx} \cup \Omega^2_{xy}$ where:

$$\Omega^2_{xx} = \left\{(x,y) \in \Omega^2 \mid x = y\right\} \quad (\Rightarrow |\Omega^2_{xx}| = 2^N)$$

$$\Omega^2_{xy} = \Omega^2 - \Omega^2_{xx} = \left\{(x,y) \in \Omega^2 \mid x \neq y\right\} \quad (\Rightarrow |\Omega^2_{xy}| = 2^N(2^N - 1))$$

Remark

$|\Omega^2_{xx}| << |\Omega^2_{xy}|$.

Secondly, we consider $\Omega^2_{xx} = A \cup C$ and $\Omega^2_{xy} = B \cup D$ where:

$$A = \left\{(x,y) \in \Omega^2_{xx} \mid \phi(x) = \mathbb{1}\right\}$$

$$B = \left\{(x,y) \in \Omega^2_{xy} \mid \phi(x) = \mathbb{1}\right\}$$

$$C = \left\{(x,y) \in \Omega^2_{xx} \mid \phi(x) \neq \mathbb{1}\right\}$$

$$D = \left\{(x,y) \in \Omega^2_{xy} \mid \phi(x) \neq \mathbb{1}\right\}$$

Definition

We say that $(x,y) \in \Omega^2$ is a configuration of type A, B, C or D, if (x,y) belongs to one of the sets A, B, C or D, respectively.

Definition

We say that, the symmetric configuration of $(x, y) \in \Omega^2$ *is the configuration* $(y, x) \in \Omega^2$*.*
In particular, the symmetric configuration of $(x, x) \in \Omega^2$ *is itself, i.e.* (x, x)*, and we will call it self-symmetric configuration.*

Definition

A limit cycle $\mathcal{C}$ *is:*

(a) *Symmetric if* $\forall (x, y) \in \mathcal{C}$, $(y, x) \in \mathcal{C}$.
(b) *Non-symmetric if* $\exists (x, y) \in \mathcal{C}$, $(y, x) \notin \mathcal{C}$.
(c) *Asymmetric if* $\forall (x, y) \in \mathcal{C}$, $(y, x) \notin \mathcal{C}$.

Remark

$\mathcal{C}$ *asymmetric* $\Rightarrow \mathcal{C}$ *non-symmetric. The converse is not necessarily true.*

Definition

$P_T \subset \Omega^2$ *is the set of configurations belonging to a limit cycle of length* $T \in \mathbb{N}$.

Example: symmetric limit cycle of length 3

$$(x,x) = \left(\begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} , \begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \right) \in C$$

$$(y,x) = \left(\begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} , \begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \right) \in B$$

$$(x,y) = \left(\begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} , \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix} \right) \in D$$

Example: non-symmetric limit cycle of length 3

$$(x,y) = \left(\begin{bmatrix} -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \right) \in D$$

$$(z,x) = \left(\begin{bmatrix} -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \end{bmatrix} \right) \in D$$

$$(y,z) = \left(\begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & 1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 \end{bmatrix} \right) \in D$$

Remark

Notice that, in fact, this limit cycle is asymmetric.

Lemma

Let x, y, z *in* Ω, *then,* $[(x,y) \rightarrow (z,x)] \Leftrightarrow [(x,z) \rightarrow (y,x)]$.
I.e., if there is a one time step transition between two configurations, then, there is also a one time step transition between their symmetric configurations, but, in the opposite sense.

Corollary

Let x^t, y^t *in* Ω, $t \in \{0, ..., q\}$, $q \in \mathbb{N}$, *then,*

$$\left(x^0, y^0\right) \rightarrow \cdots \rightarrow (x^q, y^q) \Leftrightarrow (y^q, x^q) \rightarrow \cdots \rightarrow \left(y^0, x^0\right)$$

i.e., if there is a path Q_1 *of length* q*, then, there is also a path* Q_2 *of length* q *between the symmetric configurations of* Q_1*, but, in the opposite sense (maybe* $Q_1 = Q_2$*).*

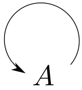
Remark

Since any configuration in the dynamic of Q2R belong to an attractor, there are two cases for the above paths Q_1 *and* Q_2*: both belong to the same limit cycle or each belong in a different one.*

Transitions between configurations of type A, B, C and D

$A = \{(x,y) \in \Omega_{xx}^2 \mid \phi(x) = \mathbb{1}\}$

$(x,x) \to (x,x)$

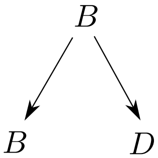


Prop.: *A is the set of fixed points of Q2R.*

$B = \left\{(x,y) \in \Omega_{xy}^2 \mid \phi(x) = \mathbb{1}\right\}$

$(x,y) \to (y,x)$

$(\Rightarrow \phi(y) = \mathbb{1} \vee \phi(y) \neq \mathbb{1})$

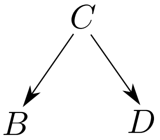


Prop. 2: $(x,y) \in P_2 \Leftrightarrow [x \neq y] \wedge [\phi(x) = \mathbb{1}] \wedge [\phi(y) = \mathbb{1}]$

$C = \{(x,y) \in \Omega_{xx}^2 \mid \phi(x) \neq \mathbb{1}\}$

$(x,x) \to (z,x)$

$(\Rightarrow \phi(z) = \mathbb{1} \vee \phi(z) \neq \mathbb{1})$

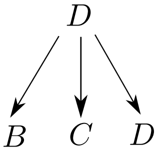


$D = \left\{(x,y) \in \Omega_{xy}^2 \mid \phi(x) \neq \mathbb{1}\right\}$

$(x,y) \to (z,x)$

$(\Rightarrow z = x \vee z \neq x)$

$(z \neq x \Rightarrow \phi(z) = \mathbb{1} \vee \phi(z) \neq \mathbb{1})$



Remark

$P_2 \subset B$. For instance, take the configuration $(\mathbb{1}, x) \in B$ where x is composed by a 2×2 block of -1s surrounded by 1s, i.e.:

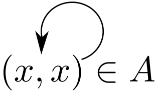
$$x = \begin{bmatrix} 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 \\ 1 & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots & 1 \\ 1 & \ddots & 1 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \cdots & 1 & -1 & -1 & 1 & \cdots & \vdots \\ \vdots & \cdots & 1 & -1 & -1 & 1 & \cdots & \vdots \\ 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \ddots & 1 \\ 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 & 1 \end{bmatrix}_{L \times L},$$

- So, $(\mathbb{1}, x) \to (x, \mathbb{1}) \to (x, x) \to (\mathbb{1}, x)$ is a limit cycle of length 3. Therefore, $(\mathbb{1}, x) \notin P_2$. Besides, this example shows that $|P_3| > 0$, for all lattice size $N = L \times L$, $L \geq 4$, L even.

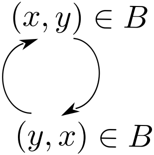
Prop. 3: Let $\{(x,y),(z,x),(y,z)\} \subset \Omega^2$ such that $(x,y) \to (z,x) \to (y,z)$. Then,

$$\{(x,y),(z,x),(y,z)\} \subseteq P_3 \Leftrightarrow \phi(x) \odot \phi(y) \odot \phi(z) = \mathbb{1}$$

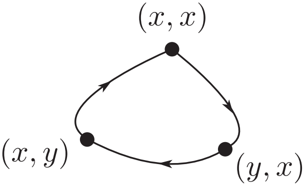
Scheme for a fixed point



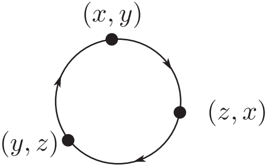
Scheme for a length-2 limit cycle



Scheme for a length-3 limit cycle (symmetric)



Scheme for a length-3 limit cycle (asymmetric)



Limit cycles with length three and higher

Corollary

Let $\mathcal{C}$ be a limit cycle of Q2R with length 3 or higher. Then:

(i) *$\mathcal{C}$ has at least one configuration of type D.*

(ii) *$\mathcal{C}$ does not have transitions of type $A \to A$, nor $B \to B$ (notice that $C \to C$ does not exist) but it could have transitions of type $D \to D$.*

(iii) *Any type D configuration comes from a type V configuration with $V \in \{B, C, D\}$.*

(iv) *If $\mathcal{C}$ has a configuration $(x, y) \in B$, then $(x, y) \to (y, x) \in D$.*

Theorem (classification of limit cycles in Q2R)

Let $\mathcal{C}$ be a limit cycle of Q2R with length $T \in \mathbb{N}$. Then $\mathcal{C}$ is of type S-I, S-II, S-III or AS, where:

- **S-I (symmetric cycle of type I).** If $T = 1$ or if there exists $p \in \mathbb{N}_0$ such that $\mathcal{C}$ has the topology of Figure (S-I), i.e., is symmetric with:

  - An odd length $T = 2(p+1)+1$.
  - Only one configuration of type C, only one configuration of type B and $(2p+1)$ configurations of type D.

- **S-II (symmetric cycle of type II).** If there exists $p \in \mathbb{N}_0$ such that $\mathcal{C}$ has the topology of Figure (S-II), i.e., is symmetric with:

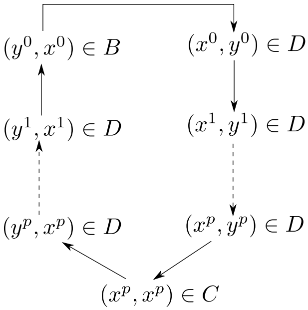
  - An even length $T = 2(p+2)$.
  - Only two configurations of type C and $2(p+1)$ configurations of type D.

- **S-III (symmetric cycle of type III).** If $T = 2$ or if there exists $p \in \mathbb{N}_0$ such that $\mathcal{C}$ has the topology of Figure (S-III), i.e., is symmetric with:

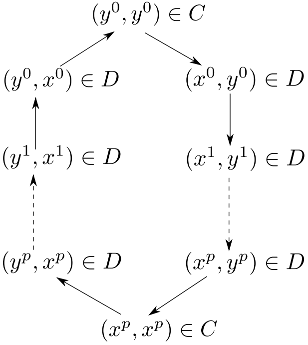
  - An even length $T = 2(p+2)$.
  - Only two type B configurations and $2(p+1)$ type D configurations.

- **AS (asymmetric cycle).** If there exists $p \in \mathbb{N} \setminus \{1\}$ such that $\mathcal{C}$ has the topology of one of the two cycles of Figure (AS), i.e., is asymmetric with:

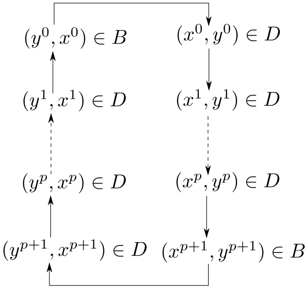
  - length $T = p+1$ (it can be even or odd, depending on the value of p).
  - All its configurations are of type D.



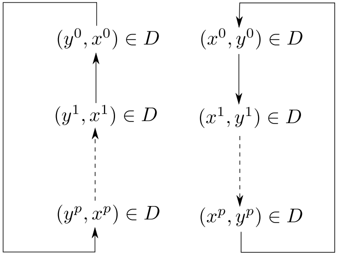
(S-I)



(S-II)



(S-III)



(AS)

Remark

Let $\mathcal{C}$ be a limit cycle of Q2R with length $T \in \mathbb{N}$ and $(x, y) \in \mathcal{C}$;

- *If T is odd, then $\mathcal{C}$ could be of type S-I or AS.*
- *If T is even, then $\mathcal{C}$ could be of type S-II, S-III or AS.*
- *If $(x, y) \in B$ then; if $\mathcal{C}$ has other configuration $(x', y') \in B$, then $\mathcal{C}$ is of type S-III, otherwise, $\mathcal{C}$ is of type S-I.*
- *If (x, y) is self-symmetric (i.e., $x = y$), then, if $\mathcal{C}$ has other self-symmetric configuration (x', y'), then $\mathcal{C}$ is of type S-II, otherwise, $\mathcal{C}$ is of type S-I.*

- The asymmetric limit cycles always appear in pairs (all the symmetric configurations of one limit cycle belongs in other limit cycle).
- $(x, y) \in \mathcal{C}$ (asymmetric) $\Rightarrow (x, y) \in D \Rightarrow [x \neq y] \wedge [\phi(x) \neq \mathbb{1}]$ (regardless the value of $\phi(y)$) but, necessarily, $\phi(y) \neq \mathbb{1}$, i.e.:

$$(x, y) \in \mathcal{C} \text{ (asymmetric)} \Rightarrow [x \neq y] \wedge [\phi(x) \neq \mathbb{1}] \wedge [\phi(y) \neq \mathbb{1}]$$

The converse relation is not necessarily true.

Other results

Definition

- $\nu(T) \equiv |P_T|$ *and* $n(T) \equiv \dfrac{\nu(T)}{T}$
- *We denote by* $\nu_{\mathrm{SI}}(T)$, $\nu_{\mathrm{SII}}(T)$, $\nu_{\mathrm{SIII}}(T)$ *and* $\nu_{\mathrm{AS}}(T)$ *as the number of configurations belonging to a limit cycle of length* T *and of type S-I, S-II, S-III and AS, respectively.*
- *Similarly,* $n_{\mathrm{SI}}(T)$, $n_{\mathrm{SII}}(T)$, $n_{\mathrm{SIII}}(T)$ *and* $n_{\mathrm{AS}}(T)$ *denote the number of limit cycles of length* T *and type S-I, S-II, S-III and AS, respectively.*

Notice that:

$$\begin{aligned}
\nu(T) &= \nu_{\mathrm{SI}}(T) + \nu_{\mathrm{SII}}(T) + \nu_{\mathrm{SIII}}(T) + \nu_{\mathrm{AS}}(T) \\
n(T) &= n_{\mathrm{SI}}(T) + n_{\mathrm{SII}}(T) + n_{\mathrm{SIII}}(T) + n_{\mathrm{AS}}(T) \\
n_q(T) &= \frac{\nu_q(T)}{T}, \quad \text{with } q \in \{\mathrm{SI}, \mathrm{SII}, \mathrm{SIII}, \mathrm{AS}\}.
\end{aligned}$$

Proposition

$$\sum_{T\geq 1}\left(n_{\mathrm{SI}}(T)+2n_{\mathrm{SII}}(T)\right)=|\Omega^2_{xx}|=2^N.$$

Corollary

$$2^{N-1}<\sum_{T\geq 1}\left(n_{\mathrm{SI}}(T)+n_{\mathrm{SII}}(T)\right)<2^N.$$

I.e., the sum of the cycles of type S-I and S-II grows exponentially with N.

Proposition (Relation between fixed points and 2-length cycles)

$\{(x,x),(y,y)\} \subseteq P_1 \Leftrightarrow \{(x,y),(y,x)\} \subseteq P_2$

Corollary

Let $x \in \Omega$ such that $\phi(x) = \mathbb{1}$. Then, in the Q2R dynamics:

- (x,x) *and* $(-x,-x)$ *are two different fixed points, as well as;*
- $(x,-x) \to (-x,x) \to (x,-x)$ *is a cycle of length 2.*

Corollary

$|P_2| = |P_1|(|P_1| - 1)$.

Staggered-states

Consider the following state $x \in \{-1, 1\}^{16}$ and its corresponding state $\phi(x) \in \{-1, 1\}^{16}$:

$$x = \begin{bmatrix} \boxed{1} & 1 & \boxed{1} & 1 \\ 1 & \boxed{-1} & 1 & \boxed{1} \\ \boxed{1} & 1 & \boxed{-1} & 1 \\ 1 & \boxed{1} & 1 & \boxed{1} \end{bmatrix} \text{ and } \phi(x) = \begin{bmatrix} 1 & \boxed{1} & 1 & \boxed{1} \\ \boxed{1} & 1 & \boxed{-1} & 1 \\ 1 & \boxed{-1} & 1 & \boxed{1} \\ \boxed{1} & 1 & \boxed{1} & 1 \end{bmatrix}$$

- The values in the boxes of x correspond to the **staggered-state** $x_B \in \{-1, 1\}^8$ while the other values that are not in boxes correspond to $x_W \in \{-1, 1\}^8$.
- Similarly, the values in the boxes of $\phi(x)$ correspond to $\phi(x)_W$ and were obtained with the values of x_B.
- The other values, that are not in the boxes of $\phi(x)$, correspond to $\phi(x)_B$ and were obtained with the values of x_W.

- **Notation:** $(\cdot) \equiv [(\cdot)_B \uplus (\cdot)_W] \in \Omega$, e.g.; $x = x_B \uplus x_W$, $\phi(x) = \phi(x)_B \uplus \phi(x)_W$, etc.
- In particular, we define the **chessboard** states:
  $\mathbb{1}_{BW} \equiv [-\mathbb{1}_B] \uplus \mathbb{1}_W$ and $\mathbb{1}_{WB} \equiv -\mathbb{1}_{BW} = \mathbb{1}_B \uplus [-\mathbb{1}_W]$

Proposition

$\forall L$ *even*, $4 \leq |P_1| < |P_2|$.

Proof. Observe that $\phi(-\mathbb{1}) = \phi(\mathbb{1}) = \phi(\mathbb{1}_{BW}) = \phi(\mathbb{1}_{WB}) = \mathbb{1}$, so:
$\{(\mathbb{1},\mathbb{1}),(-\mathbb{1},-\mathbb{1}),(\mathbb{1}_{BW},\mathbb{1}_{BW}),(\mathbb{1}_{WB},\mathbb{1}_{WB})\} \subseteq P_1$

Proposition

$\forall L \geq 4$, $|P_3| \geq 6N$.

Proof. Observe that
$\{(\mathbb{1},x),(x,\mathbb{1}),(x,x)\} \cup \{(-\mathbb{1},-x),(-x,-\mathbb{1}),(-x,-x)\} \subseteq P_3$,
where x is the state with a square of -1s.

Definition

We denote by $B_{\mathbb{1}}$ *and* $W_{\mathbb{1}}$ *as the sets of staggered-states* $x_B \in \Omega_B$ *and* $x_W \in \Omega_W$ *without null neighborhoods, respectively. That is:*

$$\begin{aligned} B_{\mathbb{1}} &= \{u \in \Omega_B : \exists x \in \Omega, [x_B = u] \wedge [\phi(x)_W = \mathbb{1}_W]\} \\ W_{\mathbb{1}} &= \{v \in \Omega_W : \exists x \in \Omega, [x_W = v] \wedge [\phi(x)_B = \mathbb{1}_B]\} \end{aligned}$$

Observe that:

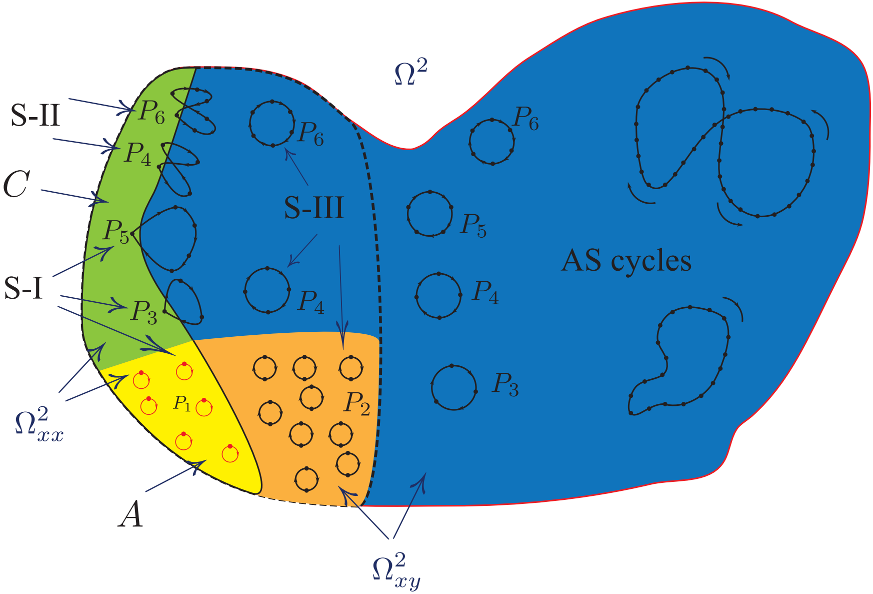
- $|B_{\mathbb{1}}| = |W_{\mathbb{1}}|$. We define $\beta \equiv |B_{\mathbb{1}}| = |W_{\mathbb{1}}|$.
- $P_1 = A = \{(x, y) \in \Omega^2_{xx} \mid \phi(x) = \mathbb{1}\}$.

Proposition

The following statements are true:

1. $\beta = 2k$, *for some* $k \in \mathbb{N}$.
2. $|P_1| = |\{x \in \Omega \mid \phi(x) = \mathbb{1}\}| = |B_{\mathbb{1}}| \cdot |W_{\mathbb{1}}| = \beta^2 = 4k^2$.
3. $|P_2| = 4k^2(4k^2 - 1)$.

General overview of the Q2R dynamics



Computations for small Q2R systems

The following Table shows the sizes of the different regions of Ω^2 for small Q2R systems.

label	variable	size	$L = 4$	$L = 6$	$L = 8$
[b]	N	L^2	16	36	64
[c]	$\lvert\Omega^2\rvert$	2^{2N}	$2^{32} \sim 4 \cdot 10^9$	$2^{72} \sim 4 \cdot 10^{21}$	$2^{128} \sim 3 \cdot 10^{38}$
[d]	$\lvert\Omega^2_{xx}\rvert$	2^N	$2^{16} = 65536$	$2^{36} \sim 7 \cdot 10^{10}$	$2^{64} \sim 2 \cdot 10^{19}$
[e]	$\lvert\Omega^2_{xy}\rvert$	[c]-[d]	$\sim 4 \cdot 10^9 <$[c]	$\sim 4 \cdot 10^{21} <$[c]	$\sim 3 \cdot 10^{38} <$[c]
[f]	|Yellow|	$\lvert P_1\rvert = \beta^2$	34^2	584^2	39426^2
[g]	| Orange|	$\lvert P_2\rvert =$ [f]([f]-1)	1335180	$\sim 10^9$	$\sim 2 \cdot 10^{18}$
[h]	|Green|	[d]-[f]	64380	$\sim 7 \cdot 10^{10} <$[d]	$\sim 2 \cdot 10^{19} <$[d]
[i]	| Blue|	[c]-([f]+[g]+[h])	$\sim 4 \cdot 10^9 <$[e]	$\sim 4 \cdot 10^{21} <$[e]	$\sim 3 \cdot 10^{38} <$[e]

Cuadro: Summary of the sizes of the main regions of Ω^2. The 1st column labels the values of the "size" column. The 2nd column has the variables and the main regions of Ω^2. In the 3rd column are the size formulas for each "variable" of the 2nd column. In the 4th, 5th and 6th columns are the calculations done for $L \in \{4, 6, 8\}$, respectively.

The 4×4 case: distribution of the configurations of Ω^2

T	$\nu_{\mathrm{SI}}(T)$	$\nu_{\mathrm{SII}}(T)$	$\nu_{\mathrm{SIII}}(T)$	$\nu_{\mathrm{AS}}(T)$	$\lvert P_T \rvert$
1	1,156	0	0	0	1,156
2	0	0	1,335,180	0	1,335,180
3	4,128	0	0	768	4,896
4	0	14,456	20,556,256	48,384,408	68,955,120
5	1,920	0	0	0	1,920
6	0	10,560	15,219,936	20,054,976	35,285,472
8	0	42,752	58,399,744	235,007,232	293,449,728
9	3,456	0	0	4,608	8,064
10	0	7,680	5,174,400	2,941,440	8,123,520
12	0	99,648	132,294,144	655,316,928	787,710,720
18	0	69,120	18,824,832	143,732,736	162,626,688
20	0	19,200	17,295,360	53,694,720	71,009,280
24	0	27,648	115,703,808	536,220,672	651,952,128
27	0	0	0	6,912	6,912
30	0	0	15,851,520	2,949,120	18,800,640
36	0	0	51,038,208	333,388,800	384,427,008
40	0	76,800	26,296,320	246,420,480	272,793,600
54	0	186,624	0	242,721,792	242,908,416
60	0	0	33,177,600	113,172,480	146,350,080
72	0	0	47,333,376	162,201,600	209,534,976
90	0	0	13,271,040	17,694,720	30,965,760
108	0	0	0	329,508,864	329,508,864
120	0	0	0	200,540,160	200,540,160
180	0	0	0	30,965,760	30,965,760
216	0	0	0	179,601,408	179,601,408
270	0	0	0	26,542,080	26,542,080
360	0	0	0	61,931,520	61,931,520
540	0	0	0	26,542,080	26,542,080
1080	0	0	0	53,084,160	53,084,160
Total	10,660	554,488	571,771,724	3,722,630,424	2^{32}

The 4×4 case: distribution of the limit cycles of Q2R

T	$n_{\mathrm{SI}}(T)$	$n_{\mathrm{SII}}(T)$	$n_{\mathrm{SIII}}(T)$	$n_{\mathrm{AS}}(T)$	$n(T)$
1	1,156	0	0	0	1,156
2	0	0	667,590	0	667,590
3	1,376	0	0	256	1,632
4	0	3614	5,139,064	12,096,102	17,238,780
5	384	0	0	0	384
6	0	1,760	2,536,656	3,342,496	5 ,880 ,912
8	0	5,344	7,299,968	29,375,904	36,681,216
9	384	0	0	512	896
10	0	768	517,440	294,144	812,352
12	0	8,304	11,024,512	54,609,744	65,642,560
18	0	3,840	1,045,824	7,985,152	9,034,816
20	0	960	864,768	2,684,736	3,550,464
24	0	1,152	4,820,992	22,342,528	27,164,672
27	0	0	0	256	256
30	0	0	528,384	98,304	626,688
36	0	0	1,417,728	9,260,800	10 ,678 ,528
40	0	1,920	657,408	6,160,512	6,819,840
54	0	3,456	0	4,494,848	4,498,304
60	0	0	552,960	1,886,208	2,439,168
72	0	0	657,408	2,252,800	2,910,208
90	0	0	147,456	196,608	344,064
108	0	0	0	3,051,008	3,051,008
120	0	0	0	1,671,168	1,671,168
180	0	0	0	172,032	172,032
216	0	0	0	831,488	831,488
270	0	0	0	98,304	98,304
360	0	0	0	172,032	172,032
540	0	0	0	49,152	49,152
1080	0	0	0	49,152	49,152
Total	3,300	31,118	37,878,158	163,176,246	201,088,822

Discussion

- Essentially, all results showed follow after the first Lemma.
- A fully classification of all Q2R attractors in 4 types of cycles consisting of symmetric and asymmetric ones was proposed.
- A general overview of the Q2R dynamic has been provided.
- Some specific results for small length cycles were proposed. For instance:
  - The total number of fixed points is of the form $|P_1| = \beta^2 = 4k^2$, with $k \in \mathbb{N}$.
  - The number of configurations belonging in a 2-length cycle is $|P_2| = \beta^2(\beta^2 - 1)$.
  - Characterization of the limit cycles with length lower or equal to 3 and related combinatorial results.
- The simple mathematical tools like the functions $\mu(\cdot)$, $\phi(\cdot)$, the Hadamard product, etc., it allows a more direct understanding of the Q2R dynamics.
- **Open problem:** a mathematical expression for β.

THANK YOU !